# Robustness evaluation of differential spectrum of integration computational algorithms

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***The impact estimates of real numbers representation inaccuracy in ECM with floating point computations on T-spectrum recurrent calculation inaccuracy of homogenous system of common linear differential equations with constant coefficients are obtained.***

***Keywords – ECM; robustness; differential transformations; integration***

Currently, numerical simulation is one of the fast-developing applications of computational facilities. In addition, use of numerical models and cloud-computing, or cluster-systems, results in significant cost-reduction of scientific and technological research. Hence, cloud-computing and cluster-systems have recently been widely used as a cheaper alternative to super-computers [1].

Now actively gaining popularity, differential transformations appear to be one of the promising methods of numerical simulation. The method of differential transformations is an applied analysis mathematical method, which makes it possible to solve integral-differential problems in numerical, analytical, and numerical-analytical forms. A possibility of recurrent calculation of differential spectrum coefficients (T-spectrum, in essence, those are Tailor's sequence coefficients) of the problem under-consideration is one of the key features of differential transformations. In this case, the calculations can be methodically easily completed in the form of corresponding ECM sub-programs, what excludes the methodical complexity of performing analytical procedures (evaluating corresponding differentials in explicit form) replacing it with computational complexity of recurrent equations. To a great extent, the above-mentioned-feature of the differential transformations method defines its possible application for solving integral-differential problems [2].

In practice, Cauchy's problem for common differential equations (CDE) system belongs to one of classes of problems that can be solved with the help of the method. Robustness of the computational schemes that are designed on a base of differential transformation approaches (and for other numerical CDE methods as well) is usually evaluated for homogenous systems of linear CDE with constant coefficients. However, in published papers the authors consider only stability characteristics of differential integration scheme; namely, transfer of integration errors from one step to the next. And in this case, T-spectrum recurrent calculation robustness analysis for every separate integration step, in other words performing differential transformation of the linear CDE system with constant coefficients, has not been carried out.

Recurrent computation of T-spectrum accounts for the lion share of mathematical operations and, as a consequence, main input into resulting computational complexity of Cauchy's problem solution for the system of CDE using differential transformations methods [2]. It is well-known that, due to finite length mantissas of the used numbers, all mathematical operations in ECM are carried out with inaccuracies of rounding-off; therefore, the numerical algorithms should be separately evaluated as to inaccuracies' impact on the final calculation result [3, 4].

Consequently, it is crucially important to research robustness of differential transformation of homogenous systems of linear common differential equations with constant coefficients, namely, estimate impact of real numbers representation inaccuracy in ECM with floating point computations on T-spectrum recurrent calculation errors of homogenous system of linear common differential equations with constant coefficients.

Cauchy's problem for homogenous system of linear CDE with constant coefficients can be described as [3]

$$\frac{du(t)}{dt} = Au(t), \quad t > 0, \quad u(t=0) = u_0 , \qquad (1)$$

where $u = (u_{j_1})_n$ – $n$-dimensional vector, with elements $u_{j_1}$; $A = (a_{j_1 j_2})_{n\times n}$ – non-singular square matrix by dimensions $n \times n$, with elements' indexes $j_1, j_2$ – line and column respectively.

Differential transformations are known as following functional transformations [2]:

$$Z(k) = \frac{h^k}{k!}\left[\frac{d^k z(t)}{dt^k}\right]_{t_*}, \qquad (2)$$

$$z(t) \approx \sum_{k=0}^{k_{\max}} \frac{(t-t_*)^k}{h^k} Z(k), \qquad (3)$$

where $t$ – argument, on which transformation is performed; $t_*$ – the argument's value, for which transformation is performed; $h$ – argument's area, where function $z(t)$ is evaluated; $k$ – integer argument $k$ = 0, 1,…; $Z(k)$ – discrete function on argument $k$; $k_{\max}$ – maximum number of T-discrete values taken into account during recovery process.

Equation (2) represents direct transformation, allowing for any original $z(t)$ to find image $Z(k)$. Inverse


978-1-xxxx-xxxx-x/17/$31.00 ©2017 IEEE

transformation, recovering original $z(t)$ in the form of Taylor's sequence section, is determined by equation (3). Multitude of values (images) $Z(k)$ has come to be known as T-spectrum, while value of the function $Z(k)$ for specific values of the argument $k$ – discrete values of T-spectrum (T-discrete values) [2].

Without losing the general view in further transformations, let us consider explicit T-scheme of integration (1) on equidistant net of calculations $\omega_i = \{t_i = ih, i = 0, 1, 2, \ldots\}$ [2, 5, 6]

$$\begin{cases} U(0) = u_i,\ t_i = ih, \\ U(k+1) = \dfrac{h}{k+1} AU(k), \text{при } k = 0, \ldots, k_{\max} - 1, \end{cases} \quad (4)$$

$$u_{i+1} = \sum_{k=0}^{k_{\max}} U_i(k), \quad (5)$$

where $u_i$ – net function, taken as the solution to (1) for $\omega_i$; $U(k)$ – T-discrete values of the solution (1) on $\omega_i$; $h$ – step of the calculation net $\omega_i$ on independent variable CDE; $k_{\max}$ – maximum number of the discrete values taken into account while recovering T-discrete value.

For the system (1), the following equation is true [7]

$$U(k_{\max}) = \frac{h^{k_{\max}}}{k_{\max}!} A^{k_{\max}} U(0), \quad (6)$$

While computing on ECM, each performed mathematical operation is accompanied by inevitable computation errors (rounding-off); consequently, the results calculated according to (6), and those according to recurrent formula (4) will not be identical, namely

$$\tilde{U}(k_{\max}) \neq U(k_{\max}), \quad (7)$$

where $\tilde{U}(k_{\max})$, $U(k_{\max})$ – T-discrete values of numerical and precise solutions (4) respectively.

Let us assess quantitative impact of this error using an approach called reverse errors analysis [3]. According to this approach, it is accepted that calculations using (4) on ECM are made precisely, but with some disturbed data.

Application of this approach to (4) leads to

$$\frac{\|\delta U(k_{\max})\|}{\|U(k_{\max})\|} \leq k_{\max} M_A^{k_{\max}} \|A^{-1}\| \|\delta A\|,\ M_A = \|A\| \|A^{-1}\|, \quad (7)$$

where $\|\cdots\|$ – norm operator (matrixes' norms compatible with those of vectors' are considered); $\|\delta U(k_{\max})\| / \|U(k_{\max})\|$ – relative error of T-spectrum recurrent computation (1); $M_A$ – condition number of matrix $A$; $\delta A$ – the disturbance that is introduced in matrix $A$ in one recurrent computation of T- discrete value.

Equation (7) shows how rounding-off errors (disturbance of matrix $A$) influence relative errors of recurrent calculations of CDE T-spectrum (1) on the base of (4).

In the vast majority of practical problems, arithmetic operations are carried out in ECM using number representation system with floating point, for which errors of representations of numbers in ECM and rounding-off errors while performing arithmetic operations (*, /, ±) can be presented as [2, 4]

$$\frac{|x - \tilde{x}|}{|x|} = |\varepsilon_x| \leq \varepsilon = \frac{1}{2} p^{-q+1}, \quad (8)$$

where $\varepsilon_x, \varepsilon$ – relative error of number $x$ representation in ECM and its maximum value respectively; $q$ – number of digits in mantissa; $p$ – the radix number of the computation system.

To quantitatively estimate $\|\delta A\|$ an approach known as error direct analysis is used [4]. According to this method the expressions under consideration are written and analyzed with account of the rules of computer arithmetic. Application of the approach mentioned above results in

$$\|\delta A\| \leq f(n)\varepsilon \|A\|, \text{ under } 3 \leq f(n) \leq n+2, \quad (9)$$

where $f(n)$ – the function that determines the coefficient of the main term of the error decomposition, depending only on the size of matrix $A$.

From (9) it is obvious that for the system of linear CDE (1) $\|\delta A_k\|$ does not depend on $k$.

With account of (9), (7) can be written as

$$\frac{\|\delta U(k_{\max})\|}{\|U(k_{\max})\|} \leq \frac{1}{2} p^{-q+1} k_{\max} f(n) M_A^{k_{\max}+1}. \quad (10)$$

The right part of (10) is an increasing function from the maximum number of T-discrete values – $k_{\max}$. So, robust calculation of CDE T-spectrum (1) (performing calculations with satisfactory precision) is possible only for a limited number $k_{\max}$, namely

$$k_{\max}^* = \arg\left\{1 \geq \max\left(\frac{1}{2} p^{-q+1} k_{\max} f(n) M_A^{k_{\max}+1}\right)\right\}, \quad (11)$$

where $p$ – the radix; $q$ – number of digits in mantissa; $k_{\max}$ – maximum number of the discrete values while recovering T-discrete value; $f(n)$ – the function that determines the coefficient of the main term of the error decomposition (9); $M_A$ – condition number of matrix $A$.

Results of calculation, with account of (11), of maximum number of T-discrete values for the system of linear CDE with constant coefficients, which can be robustly (with acceptable accuracy) calculated, for radix number $p = 10$ are presented in table 1, where $k_{\max}^*$ – the number of senior T-discrete values calculated with acceptable precision, $q$ – the number of mantissa orders, $M_A$ – condition number, $n$ – dimension of CDE (in the first column the value under $f(n) = 3$, and in the second - under $f(n) = n + 2$ ).

Analysis of the data presented in table. 1 shows:

robustness (precision) of recurrent calculation of T-spectrum of homogenous system of linear CDE with constant coefficients substantially depends upon mantissa length of the numbers used in the ECM and condition number of the matrix in the right part of CDE equation;

under small mantissa length and big condition number of the matrix in the right part of the homogenous system of linear CDE with constant coefficients, it is impossible to exactly calculate a big number of T-discrete values for such a CDE.

With account of above, a definition is introduced – a numerical calculation robustness area of T-spectrum of homogenous system of linear CDE with constant coefficients, which is determined as the set of all numbers of recurrently calculable CDE T-discrete values of type (1), and for which condition (11) is observed.

TABLE I. MAXIMUM NUMBER OF ROBUSTLY CALCULABLE T-DISCRETES OF CDE SYSTEM

| $q$ | $M_A$ | $k^*_{\max}$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $n=3$ | | $n=10$ | | $n=20$ | |
| 10 | $10^1$ | 6 | 6 | 6 | 6 | 6 | 6 |
| | $10^2$ | 3 | 3 | 3 | 2 | 3 | 2 |
| | $10^3$ | 1 | 1 | 1 | 1 | 1 | 1 |
| | $10^5$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | $10^1$ | 11 | 11 | 11 | 11 | 11 | 10 |
| | $10^2$ | 5 | 5 | 5 | 5 | 5 | 5 |
| | $10^3$ | 3 | 3 | 3 | 3 | 3 | 3 |
| | $10^5$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 20 | $10^1$ | 16 | 16 | 16 | 16 | 16 | 15 |
| | $10^2$ | 7 | 7 | 7 | 7 | 7 | 7 |
| | $10^3$ | 5 | 4 | 5 | 4 | 5 | 4 |
| | $10^5$ | 2 | 2 | 2 | 2 | 2 | 2 |
| 25 | $10^1$ | 21 | 21 | 21 | 20 | 21 | 20 |
| | $10^2$ | 10 | 10 | 10 | 10 | 10 | 9 |
| | $10^3$ | 6 | 6 | 6 | 6 | 6 | 6 |
| | $10^5$ | 3 | 3 | 3 | 3 | 3 | 3 |
| 30 | $10^1$ | 26 | 26 | 26 | 25 | 26 | 25 |
| | $10^2$ | 12 | 12 | 12 | 12 | 12 | 12 |
| | $10^3$ | 8 | 8 | 8 | 8 | 8 | 8 |
| | $10^5$ | 4 | 4 | 4 | 4 | 4 | 4 |

It is necessary to separately stress the difference between the obtained results on robustness of CDE T-spectrum numerical calculation (1) and stability provisions of T-schemes of Cauchy's problem solution for CDE (and their systems).

Computation scheme (4) and (5) consists of the combination of T-discrete values, and represents a differential equation defining solution for CDE in the nodes of the calculation net. This scheme permits to obtain a solution for CDE in the node $t_{i+1}$ from its value in the previous node $t_i$. While assessing robustness of T-scheme of CDE solution, the properties of the obtained differential equation on the introduced calculation net are considered with respect to the impact (distribution) of CDE numerical solution error in the node $t_i$ on the error of CDE numerical solution in the next node $t_{i+1}$. Consequently, some limitations are imposed on the maximum integration step $h \le h_{\max}(A, k_{\max})$ with account of CDE features (matrix $A$) and T-scheme parameters (the number of T-discrete values – $k_{\max}$). In this case, the greater $k_{\max}$, the less limitations are imposed on the step $h_{\max}$.

Taking into account notations introduced above, it is possible to argue that robustness assessment of numerical calculation of CDE T-spectrum (1) considers calculation of T-scheme elements in one node of the calculation net and evaluates errors transfer from numerical calculation T-discrete value $k$ onto those of T-discrete value $k+1$. The results of such a research, with account of the CDE properties (matrix $A$) and the features of numerical notation in ECM, are the limitations imposed on T-discrete values maximum number – $k_{\max}$, which is calculated with satisfactory precision.

From all above, it is clear that robustness assessment of CDE T-spectrum numerical calculation (1) and stability provisions of CDE solution with a help of T-scheme consider different aspects of differential transformations applications.

**Conclusion.** Obtained equations (10) and (11) represent an estimate of T-spectrum calculation accuracy for homogenous system of linear CDE with constant coefficients according to majorizing assessment of equivalent disturbances' norms; therefore, such estimates are majorizing [4].

With account of obtained majoring accuracy estimates of T-spectrum calculation of homogenous system of linear CDE with constant coefficients, the new definition is introduced - the area of robust numerical calculation of T-spectrum of homogenous system of linear CDE with constant coefficients. Applying this definition, that maximum number of T-discrete values of homogenous system of linear CDE with constant coefficients can be calculated, which can be robustly (with acceptable accuracy) determined with account of the used ECM notation system features and the condition number of the matrix in the right part of linear CDE system under consideration.

For CDE systems of a broader type, estimates (10) and (11) and, correspondingly, their areas of stability will not surpass obtained estimates for homogenous system of linear CDE with constant coefficients. This is because, according to the properties of differential transformations, the number of arithmetic operation to calculate T-spectrum of such a CDE will be much greater, and, consequently, so will be the introduced errors. This leads to the conclusion that robustness area of such CDE systems will be less than for homogenous system of linear CDE (1).

In general, obtained results permit to effectively use differential transformations for developing numerical models, which can be calculated either in cluster systems or with the help of cloud technologies [5].


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